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A strength reliability model of unidirectional fiber-reinforced ceramic matrix composites by Markov process *

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Abstract—We propose a stochastic process analysis for predicting the strength and reliability of a unidirectional fiber-reinforced ceramic matrix composite. The analysis is based on a Markov process, in which it is assumed that a state of damage in the composite is developed with each fiber breakage. When the Weibull distribution is used to describe the strength distribution of the fiber, the probability of being in each state can be solved analytically in a closed form. Using the solutions of the probabilities, a discussion about damage tolerance of the composites is quantitatively developed, from the viewpoint of materials reliability engineering. To compare the proposed stochastic process analysis with previously proposed strength models in the basis of classical bundle theory, we obtained the expected value and variance in the composite stress from solutions of the probabilities of being in the states. The expected value and variance both consist of two terms, equivalent to the effects of both bundle structure and stress recovery in broken fibers. These are surprisingly in agreement with the solutions analyzed by Phoenix and Raj [9]. The effect of stress recovery in broken fibers produces a positive in the expected value and a negative in the variance and is thus a significant mechanism for increasing the strength and reliability of the composite. In addition we predicted the expected values of composite strengths and variances from the solutions. The corresponding value of normalized fiber stress to obtain the expected value in strength and the variance agreed relatively well with the value predicted by Hui *et al.* [13]. We further verified that the composite strength obeys a normal distribution, which has the expected value and standard deviation predicted above. Finally, we predicted the probabilities in strength of the composites with various sizes and concluded that the ceramic matrix composite is quite reliable in strength when the number of fibers corresponds to that in practical use.

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1. INTRODUCTION

Ceramic matrix composites reinforced with continuous ceramic fibers are highly valued as key-materials for application to high-temperature structural components because their low toughness, a weak point inherent in monolithic ceramics, has been dramatically improved by a fiber/matrix sliding behavior. When the composite is loaded along the fiber-axis, matrix cracking first occurs at strains exceeding the matrix fracture strain. Although the matrix crack propagates across the entire cross-section of the composite, the fibers are not broken due to debonding and sliding at the relatively weak interfaces between them and the matrix. The matrix stress lost due to cracking recovers from zero at the crack plane because of the interfacial sliding friction. As this phenomenon is exhibited repeatedly, except in stress recovery regions, multiple matrix cracks occur. These matrix cracks will saturate at some stress level when the distance between two neighboring matrix cracks does not suffice to cause the necessary stress between them. Also, matrix cracks occur at composite strains far lower than strains at which significant fiber breaks occur in the composite. If the applied load is further increased, next the fibers start to break. Broken fibers also recover in stress due to the same mechanism. Although in this situation the composite behaves inelastically, the fibers can sustain further an additional applied load. Thus, papers are often focused on the post-matrix crack saturation behavior of the composite and its tensile strength [1–4].

Following matrix cracking, the axial fiber stress changes from the maximum at a point between matrix crack surfaces to a minimum, halfway between adjacent matrix cracks. The repeatedy ocurring multiple matrix cracking brings quasi-periodic distributions to both the fiber and matrix stresses, respectively [5]. In this situation the matrix stress is largely reduced due to the multiple crackings, as verified by the finite element calculation [6]. Thus, the quasi-periodic matrix stress is neglected in the modeling of composite strength; uniform fiber stress is assumed. Nevertheless, analytical results without any matrix stress agree well with measured experimental values, as shown in the literature [1, 5, 7]. On the other hand, debonding and sliding at the interfaces in the composite prevent stress redistribution from local stress concentration around broken fibers. Then, the loads lost due to fiber breaks will be shared globally among surviving fibers with an increase in strain; a global fiber stress distribution is also verified by the same finite element calculation as the above [6]. Assuming uniform fiber stress distribution and no interaction among fibers, the composite stress and strength are often expressed on the basis of classical bundle theory. A different point from the bundle theory is the point that stress on broken fibers in the composite is able to recover at regions far from their breaking points. The structure of the composites is then regarded as a chain-of-bundles [8], in which unit fiber bundles are linked with each other in series as a chain. The composite strength is determined by the weakest of all unit fiber bundles, similarly to the weakest link theory. Using the chain-of-bundles, Phoenix *et al.* [9, 10] clarified size effects in the distribution of composite strength, into which the mean and variance of the bundle theory were incorporated mathematically and reasonably. As estimated easily, the chain-of-bundles structure allows strain localization, because each of the unit fiber bundles would contain a different number of fiber beaks. Curtin [11, 12] considered this influence analytically and through Monte-Carlo simulation; he concluded that strain localization sets in only after the maximum strength is achieved. Thus, in spite of the limited assumptions as mentioned above, not only tensile strength of ceramic matrix composites including the size effects, but other properties such as fiber pullout length [11], composite toughness [12], exact solution for fiber fragmentation [13], and fracture process by Monte-Carlo simulation [14] have been clarified for the past one-and-a-half decades and have contributed to development of the area of composite mechanics.

On the other hand, for structural materials subject to cyclic or creep load, their stochastic nature related with damage tolerant design has often been discussed, from the viewpoint of materials reliability engineering. For example, fatigue crack propagation in metal is often treated as a temporal stochastic process [15]. In the above papers regarding ceramic matrix composites, however, a stochastic view of damage evolution process in the composite with an increase in stress or with time has never been included. We understand that, in a ceramic matrix composite, fiber breaks and matrix cracks occur randomly, and accumulate discretely in the composite with an increase in stress or with time. In other words, the amount of damage evolving in the composite, such as the accumulation of fiber breaks, varies stochastically. Figure 1 shows a schematic of possible stochastic damage evolution in a composite. Sample 1 shown by the solid line achieves a critical damage quantity at the stress σ_2, earlier than sample 2 shown by the dotted line, even though sample 1 indicates a smaller damage quantity at stress σ_1. Such a transition in damage accumulation may obviously be regarded as a stochastic process, and is

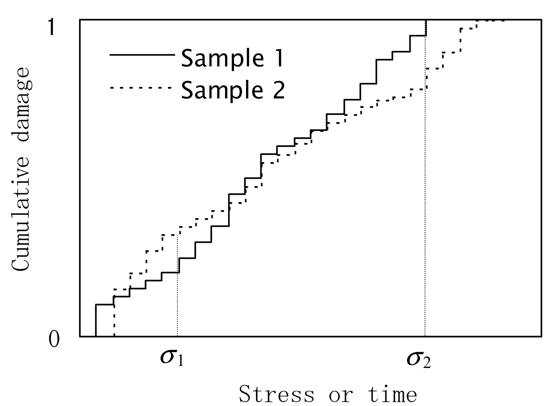

Figure 1. A schematic of damage accumulation in a ceramic matrix composite.

a possible process to be brought into calculations for the composite. The purpose of this study is therefore to incorporate a Markov process [16] into the fiber breaking process in a unit fiber bundle of the chain-of-bundles. The process is assumed to be a pure birth process, and described with simultaneous first-order differential equations. Then, the probability of being in each state of this process was obtained analytically. In addition, we predicted the expected values and variances of both the number of fiber breaks and the composites stress from the analytical solutions. These statistical parameters agree exactly with the parameters obtained by Phoenix and Raj [9]. We then approximated the maximum stress of the expected value of the composite stress and its corresponding standard deviation, and proved that these can be treated as a normal distribution. Finally, we concluded that a composite with a large number of fibers and large length approaches to a unit fiber bundle in strength, despite the fact that the composite is modeled as a chain-of-bundles.

2. MARKOV PROCESS MODEL FOR A CERAMIC MATRIX COMPOSITE

2.1. Solutions for probabilities of being in states

This section proposes a Markov process model to predict the strength and reliability of a ceramic matrix composite based on the following assumptions:

(1) Interfacial bonds between the fibers and matrix of the composites are weak, so that interfacial debonding occurs with premature matrix cracking. Hence, a load applied to the composites is sustained by only the fibers after matrix crack saturation. No matrix cracks interact with fibers [5, 7, 9–12].

(2) The stress acting on broken fibers recovers to the stress level of intact fibers in regions away from the broken point, because of interfacial sliding friction. A length of twice the stress recovery length, called the 'ineffective length' is a parameter of the composite. Then, the entire structure of the composite consists of a series of unit fiber bundles, each of which has the ineffective length, called the 'chain-of-bundles' [8, 17–19].

(3) The global load-sharing rule is assumed in this model. This means, there is no local load redistribution around broken fibers in the composite, and the fiber stress increases with uniform distribution and without any stepped-stress on intact fibers [5, 7, 9–12].

The significance of assumption (2) is to extend a limiting constant length of the reference (Goda, 2002) to the ineffective length, which is changed with increased stress. In other words, the previously proposed model is extended to the chain-of-bundles structure with a longer length. Assumptions (1) and (3) are based on Curtin's model [5], which were comparable to the results of the FEM analysis, as verified in the reference [6].

The material analyzed in this study is a ceramic matrix composite consisting of N fibers without any initial breakage point. When a load is applied along its axial

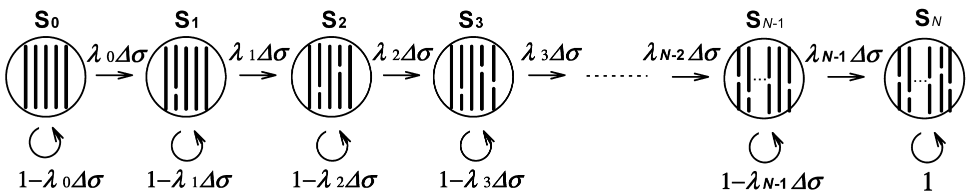

Figure 2. State transition diagram of fiber breaking process.

direction, multiple cracks occur in the matrix and each fiber sustains the entire load with an even fiber stress σ. When a higher load is applied to the composite, the fibers start to break. We consider here that each fiber is fractured in accordance with a stochastic process in a discrete state space. That is to say, we express the fiber breaking process as a Markov process, in which a damage state transits with a failure rate to a larger damage state. The failure rate is often treated as a function of time t, but in this study this rate was assumed as a function of stress, given as $\sigma = \alpha t$ (α: a positive stress rate). Figure 2 shows the state transition diagram used in this analysis, in which all damage states regarding the fiber breaking process are set. This setting will enable us to predict, probabilistically, a damage state in the composite at every stress. Also, this state transition diagram is characterized by a forward transition process of the states, i.e. the pure birth process, because the fiber breaking process cannot proceed reversibly. In the figure, S_i ($i = 0, 1, \dots, N$) denotes the state consisting of i broken fibers and $(N - i)$ intact fibers, and λ_i ($i = 0, 1, \dots, N - 1$) denotes the failure rate from the state S_i to S_{i+1}, which is a power type of function, as mentioned later. The probability that at least the weakest fiber is broken in a stress increment $\Delta\sigma$ from a stress σ, is $\lambda_0\Delta\sigma + O(\Delta\sigma)$, which is the probability of the transition from the state S_0. Here, $\lambda_0\Delta\sigma$ is the probability of the transition from S_0 to S_1, and the residue term, $O(\Delta\sigma)$, means the probability of the transition from S_0 to S_2 or more. In general, the transition probability from S_i, the probability that at least the $(i+1)$-th weakest fiber is broken in $\Delta\sigma$, is expressed as $\lambda_i\Delta\sigma + O(\Delta\sigma)$. Finally, the process enters state S_N called the 'absorbing state', in which all the fibers are broken. When $\Delta\sigma \to 0$, $O(\Delta\sigma)$ approaches to zero and the fiber breaking process can be expressed with ordinary simultaneous differential equations as follows:

$$
\begin{aligned}
&\frac{\mathrm{d}P_0}{\mathrm{d}\sigma} + \lambda_0 P_0 = 0,\\
&\frac{\mathrm{d}P_1}{\mathrm{d}\sigma} - \lambda_0 P_0 + \lambda_1 P_1 = 0,\\
&\cdots\cdots\\
&\frac{\mathrm{d}P_{N-1}}{\mathrm{d}\sigma} - \lambda_{N-2} P_{N-2} + \lambda_{N-1} P_{N-1} = 0,\\
&\frac{\mathrm{d}P_N}{\mathrm{d}\sigma} - \lambda_{N-1} P_{N-1} = 0,
\end{aligned}
\tag{1}
$$

where, $P_0, P_1, \ldots, P_{N-1}$ and P_N are the probabilities of being in the states $S_0, S_1, \ldots, S_{N-1}$ and S_N. When $\lambda_j(\sigma) = h_j\lambda_0(\sigma)$ and h_j $(j = 0, 1, 2, \ldots, N-1)$ are constants, the solutions of equation (1) are inductively obtained as:

$$P_i = (-1)^i \sum_{j=0}^{i} \left(\prod_{\substack{k=0 \\ k\neq j}}^{i} \frac{h_k}{h_j - h_k} \right) \frac{h_j}{h_i} \exp\{-h_j \Lambda_0(\sigma)\} \quad (i = 0, 1, 2, \ldots, N-1), \tag{2}$$

and

$$P_N = 1 - (-1)^{N-1} \sum_{j=0}^{N-1} \left(\prod_{\substack{k=0 \\ k\neq j}}^{N-1} \frac{h_k}{h_j - h_k} \right) \exp\{-h_j \Lambda_0(\sigma)\}, \tag{3}$$

where $\Lambda_0(\sigma)$ is the primitive function of λ_0. In obtaining the solutions, equations (2) and (3), the following initial and converging conditions were used:

$$P_0 = 1 \quad \text{and} \quad P_i = 0 \quad (i = 1, 2, \ldots, N) \quad \text{for } \sigma = 0, \tag{4}$$

$$P_N = 1 \quad \text{for } \sigma \to \infty. \tag{5}$$

As easily demonstrated, when the stress rate α is changed to zero or becomes negative during loading, no state can transit to the next state. Since stress exhibits the nature of a tensor, the use of a stress variable in this analysis implies the possibility that the Markov process model can be extended to a multi-axial stress state.

The next problem of interest is how to derive the failure rate $\lambda_j(\sigma)$. As mentioned in the Introduction, broken fibers in a ceramic matrix composite recover in stress due to sliding resistance at the interfaces between the fibers and matrix. If the sliding resistance is given as a constant τ, the stress recovery length $\delta(\sigma)$ is expressed as [5]:

$$\delta(\sigma) = \frac{r}{2\tau}\sigma, \tag{6}$$

where r is the radius of the single fiber. Equation (6) can be used for analyses, if the composite length is sufficiently longer than $\delta(\sigma)$. As mentioned in assumption (2), the effect of fiber breakage may be limited within the ineffective length, so that the fiber strength is given within the ineffective length, i.e. $2 \times \delta(\sigma)$. Thus, the fiber strength was assumed to obey the two-parameter Weibull distribution [9] as follows:

$$F(\delta(\sigma), \sigma) = 1 - \exp\left\{ -\frac{2\delta(\sigma)}{L_0} \left(\frac{\sigma}{\sigma_0} \right)^m \right\}, \tag{7}$$

where m and σ_0 are the Weibull shape and scale parameters, respectively. L_0 is a standard gauge length at which the Weibull parameters are estimated. The failure rate λ_f of this distribution function is given as:

$$\lambda_f = \frac{f(\delta(\sigma), \sigma)}{R(\delta(\sigma), \sigma)} = \frac{2\delta(\sigma)}{L_0}\frac{m\sigma^{m-1}}{\sigma_0^m} + \frac{2}{L_0}\frac{\mathrm{d}\delta(\sigma)}{\mathrm{d}\sigma}\left(\frac{\sigma}{\sigma_0}\right)^m \equiv \lambda_\sigma + \lambda_\delta, \tag{8}$$

where $f(\delta(\sigma), \sigma)$ and $R(\delta(\sigma), \sigma)$ are, respectively, the probability density and reliability of equation (7). λ_σ is the failure rate due only to the stress increase without any increase in the ineffective length, that is,

$$\lambda_\sigma = f(\delta = \delta_{const.}, \sigma)/R(\delta = \delta_{const.}, \sigma).$$

In contrast, λ_δ is the failure rate due to the increase in the ineffective length. The increment of the failure rate caused by the increment of a newly extended length, $d\delta(\sigma) \times 2$, is given as

$$\frac{\mathrm{d}\delta(\sigma) \times 2}{L_0}\left(\frac{m\sigma^{m-1}}{\sigma_0^m}\right).$$

λ_δ is obtained by integrating the above term at the stress interval $[0, \sigma]$. The ratio of λ_σ to λ_δ interests us:

$$\frac{\lambda_\sigma}{\lambda_\delta} = m. \tag{9}$$

Equation (9) means many fibers break with an increase in fiber stress, but as m is decreased, some breaks are caused by increasing the ineffective length. Also equation (9) follows only if equation (6) is valid. In the fiber breaking process of this composite, two normalizing scales have been used [5, 9]:

$$\delta_c = \left(\frac{\sigma_0 r L_0^{1/m}}{\tau}\right)^{\frac{m}{m+1}}, \tag{10}$$

for length, and

$$\sigma_c = \left(\frac{\sigma_0^m \tau L_0}{r}\right)^{\frac{1}{m+1}}, \tag{11}$$

for stress. These equations (10) and (11) give the relation similar to equation (6) as:

$$\delta_c = \frac{\sigma_c r}{\tau} = 2\delta(\sigma_c).$$

Substitution of equation (11) into equation (7) yields

$$F(\sigma) = 1 - \exp\left\{-\left(\frac{\sigma}{\sigma_c}\right)^{m+1}\right\}. \tag{12}$$

Equation (12) is a two-parameter Weibull distribution with a scale parameter equal to the normalized stress σ_c. The failure rate λ_f becomes:

$$\lambda_f = \frac{(m+1)\,\sigma^m}{\sigma_c^{m+1}}. \tag{13}$$

Since weaker fibers are broken first due to the global load sharing, as mentioned in assumption (3), the number of intact fibers $(N-i)$ is a possible number transiting to the next state. Thus, the failure rates λ_i mentioned earlier are obtained by multiplying λ_f by $(N-i)$ as follows:

$$\begin{aligned}\lambda_i &= (N-i)\lambda_f \\ &= \frac{N-i}{N}\lambda_0 \equiv h_i\lambda_0 \quad (i = 0, 1, 2, \ldots, N-1).\end{aligned} \tag{14}$$

Equation (14) means that failure rates in the fiber breaking process of a ceramic matrix composite may be applied for the forms of equations (2) and (3). By substituting equation (14) into equations (2) and (3), the solutions for $P_0, P_1, \ldots, P_N$ are rewritten as follows:

$$P_i = (-1)^i \binom{N}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \exp(-\Lambda_j), \quad (i = 0, 1, \ldots, N-1), \tag{15}$$

and

$$P_N = 1 - (-1)^{N-1} \sum_{j=0}^{N-1} (-1)^j \binom{N}{j} \exp(-\Lambda_j), \tag{16}$$

where

$$\binom{s}{t} = \frac{s!}{(s-t)!t!} \text{ for integers } s, t, \text{ and } \Lambda_i = (N-i)\left(\frac{\sigma}{\sigma_c}\right)^{m+1}.$$

Thus, the probability of being in each state is determined analytically, by using the Markov process model.

2.2. Calculation results

As an example of the fiber breaking process, we calculated the probabilities $P_0, P_1, \ldots, P_N$ for $N = 20$. The results are shown in Fig. 3, in which P_1, P_3, $P_5, \ldots, P_{19}$ have been omitted for simplicity. As seen in the change of P_0, no fiber is broken until around $\sigma/\sigma_c = 0.20$, after which the fibers start to break with a further increase in σ/σ_c. First, P_2 increases, and secondly, P_4 increases. P_2 peaks around $\sigma/\sigma_c = 0.65$ then decreases. P_4 also peaks and then decreases. This fiber breakage state evolves with increasing σ/σ_c, and reaches a state in which there are more fiber breaks. Finally, the state S_{20}, in which all the fibers are broken, begins to appear, and reaches $P_{20} = 1$ around $\sigma/\sigma_c = 1.50$. Thus, the fiber breaking process

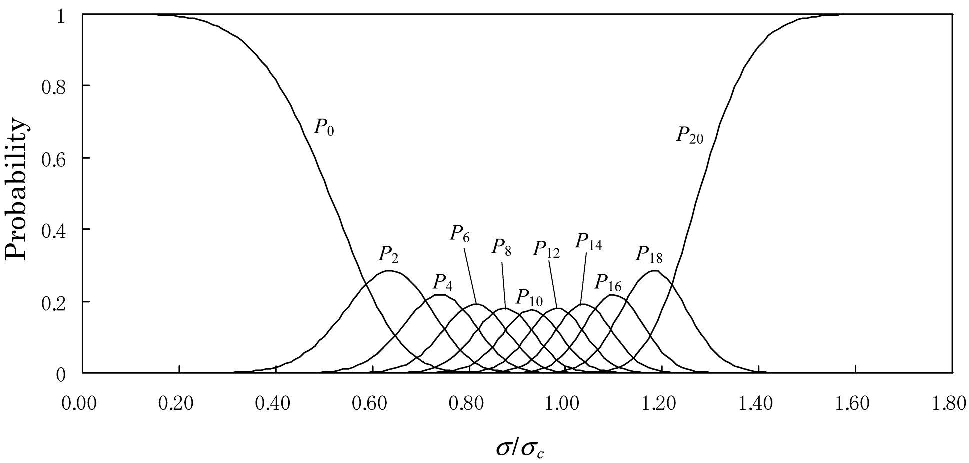

Figure 3. Probabilities of being in states obtained from equations (15) and (16) ($m = 4$, $N = 20$).

of a ceramic matrix composite can be estimated stochastically by using the proposed Markov process model.

As the number of fibers is increased, the number of states is increased and the probabilities of being in the states are decreased. In this situation we may consider analyzing the probability of being in one of several states. That is, the probability of being in one of the states S_0 to S_k, denoted as $P_{0,k}$, is given as:

$$\begin{aligned} P_{0,k} &= P_r\{S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_k\} \\ &= \sum_{i=0}^{k} P_i \\ &= \sum_{i=0}^{k} (-1)^i \binom{N}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \exp(-\Lambda_j) \\ &= \sum_{j=0}^{k} (-1)^j \binom{N}{i} \exp(-\Lambda_j) \sum_{i=j}^{k} (-1)^i \binom{N-j}{i-j} \\ &= \sum_{j=0}^{k} (-1)^{k+j} \binom{N}{j} \binom{N-j-1}{k-j} \exp(-\Lambda_j). \end{aligned} \tag{17}$$

For example, when we choose $N = 100$ and $k = 5$, $P_{0,k}$ means the probability that 5% or less of the fibers is broken. From equation (17) the probability of being in one of the states S_l to S_k, denoted as $P_{l,k}$, is also given as:

$$P_{l,k} = P_{0,k} - P_{0,l}. \tag{18}$$

As seen in Fig. 1, the damage accumulation may be defined as a damage variable varying from 0 to 1. In developing a damage tolerance design for the composites, the damage variable may be adopted as an effective design parameter. In the above example of $N = 100$ and $k = 5$, the damage variable is 0.05, and an applied stress giving 0.05 may be used as the design stress. The meaning of the probability $P_{0,k}$ is thus that design stress can be treated from the standpoint of reliability engineering. Figure 4 shows probabilities $P_{0,5}$, $P_{0,10}$ and $P_{0,20}$, for $N = 100$, predicted from equation (17). Probability $P_{0,5}$ is distributed approximately from stress 0.45 to

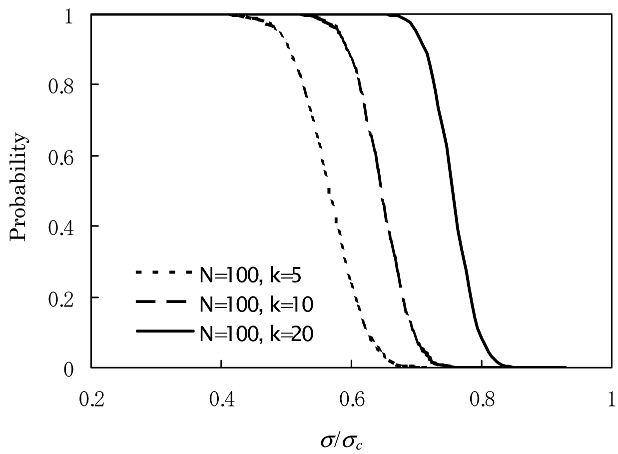

Figure 4. Probabilities, $P_{0,k}$ of being in one of the states S_0 to S_k.

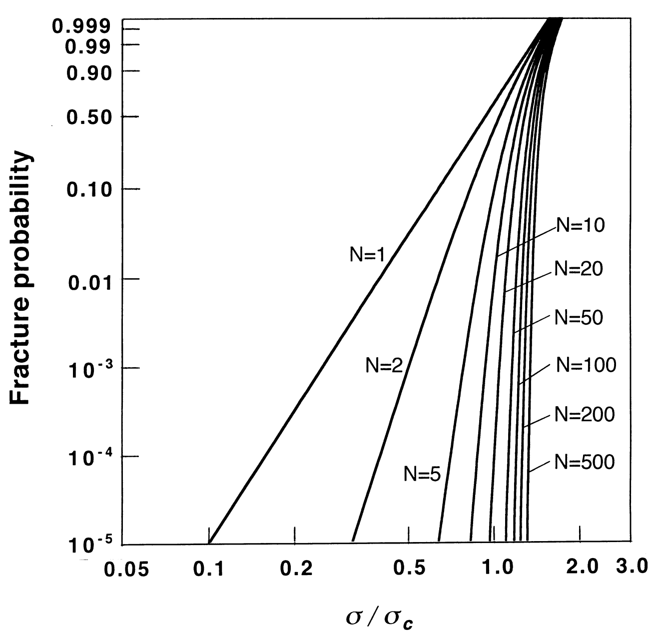

Figure 5. Fracture probabilities of all the fibers in a ceramic matrix composite predicted by Markov process model ($m = 4$).

0.65. As k is increased, the probability shifts to a larger normalized stress. For example, when we take $P_{0,5} = 0.95$ as a reliability index, the normalized stress is given as $\sigma/\sigma_c = 0.48$. Thus, the proposed Markov process model enables us to treat, quantitatively, fiber damage tolerance of this material from the viewpoint of materials reliability engineering.

From the above solutions we can also estimate the distribution for the composite's fracture stress. Figure 5 shows Weibull plots of the probabilities P_N (predicted from equation (16)) for the composites with various numbers of fibers. P_N is not the probability at maximum stress of the composite, but the fracture probability of all the fibers. As shown in the figure, the fracture probability shifts to a higher stress with an increase in the number of fibers. Furthermore, the slope becomes larger and reaches almost vertical when $N \geqslant 50$. This means the fracture stress of the composite with a large number of fibers does not vary stochastically. Thus, the stress level at all fiber breaks may be treated as a deterministic level when $N \geqslant 50$.

3. EXPECTED VALUE AND VARIANCE IN COMPOSITE STRESS

3.1. Analytical solutions

The composite can stochastically be in various states at a stress σ, so that the number of broken fibers is regarded as a random variable. To know the expected value and variance of the number of broken fibers is quite important in characterizing the damage state of the composite. In the present analysis, thus, the expected value and variance was analytically predicted. The expected value $E\{i\}$ of the number of broken fibers, i, at stress σ is then given as (see Appendix 1):

$$\begin{aligned} E\{i\} &= \sum_{i=0}^{N} i P_i \\ &= \sum_{i=0}^{N} N P_i - \sum_{i=0}^{N} (N-i) P_i = N \sum_{i=0}^{N} P_i - \sum_{i=0}^{N-1} (N-i) P_i \\ &= N - N \exp(-\Lambda_f), \\ &\quad \left(\because \sum_{i=0}^{N} P_i = 1 \right) \\ &= N \left(1 - e^{-\Lambda_f} \right). \end{aligned} \tag{19}$$

Since the term $\sum_{i=0}^{N-1} (N-i) P_i$ is the expected value of the number of intact fibers, equation (19) can be also written as:

$$E\{i\} = N - E\{N-i\}, \tag{19'}$$

where, Λ_f is the primitive function of λ_f, i.e. $\Lambda_f = (\sigma/\sigma_c)^{m+1}$. As known from equation (19), the expected value is the product of the number of fibers and the fracture probability of fibers. The variance $V\{i\}$ is given as (see Appendix 2):

$$
\begin{aligned}
V\{i\} &= \sum_{i=0}^{N} i^2 P_i - (E\{i\})^2 \\
&= \sum_{i=0}^{N} \left\{(N-i)^2 P_i\ N^2 P_i + 2Ni\right\}(E\{i\})^2 = \sum_{i=0}^{N} (N\ i)^2 P_i\ (E\{N\ i\})^2 \\
&= N e^{-\Lambda_f}\left(1 - e^{-\Lambda_f}\right).
\end{aligned} \tag{20}
$$

From equation (20), the variance of the number of broken fibers is equal to that of the number of intact fibers, i.e. $V\{i\} = V\{N - i\}$. Once the expected value and variance of the number of broken fibers are known, the expected value and variance of the nominal stress, σ_n, of the composite can also be predicted.

According to the assumption (2), the nominal stress of the composite in the state S_i is given as:

$$\sigma_n = V_f \sigma_f = V_f \left\{\frac{N-i}{N}\sigma + \frac{i}{N}\frac{2\tau}{r}\langle L(\sigma)\rangle\right\}, \tag{21}$$

where σ_f is the nominal stress of fibers, and $\langle L(\sigma)\rangle$ indicates an average distance from crack planes in the matrix to a fiber breakage point within the ineffective length. Equation (21) consists of the sum of the stresses on intact and broken fibers. Substitution of equation (21) into the approximation $\langle L(\sigma)\rangle/\delta(\sigma) \approx 1/2$ [5], yields:

$$\sigma_n = V_f \left\{\frac{N-i}{N}\sigma + \frac{i}{N}\frac{\sigma}{2}\right\}. \tag{22}$$

Then, the expected value $E\{\sigma_n\}$ of σ_n is given as (see Appendix 3):

$$E\{\sigma_n\} = V_f\left\{\sigma e^{-\Lambda_f} + \frac{\sigma}{2}\left(1 - e^{-\Lambda_f}\right)\right\}. \tag{23}$$

The first term $\sigma e^{-\Lambda_f}$ in the right side of equation (23) agrees with the expected value of the well-known dry bundle theory [20, 21]. The second term $\sigma(1 - e^{-\Lambda_f})/2$, therefore, means a 'contribution' term brought from stress recovery behavior in broken fibers. As the term $(1-e^{-\Lambda_f})$ is in the range of 0 to 1, the term $\sigma(1-e^{-\Lambda_f})/2$ must be in the range of 0 to $\sigma/2$. In other words, the value $E\{\sigma_n\}/V_f$ is larger than the expected value of a dry bundle. Thus, we have proven that stress recovery behavior due to sliding resistance is an effective mechanism to increase the average of the composite stress. The variance $V\{\sigma_n\}$ of the nominal stress σ_n is given as (see Appendix 4)

$$V\{\sigma_n\} = V_f^2\left\{\frac{\sigma^2}{N}e^{-\Lambda_f}\left(1 - e^{-\Lambda_f}\right) - \frac{3}{4}\frac{\sigma^2}{N}e^{-\Lambda_f}\left(1 - e^{-\Lambda_f}\right)\right\}. \tag{24}$$

The first term $\sigma^2 e^{-\Lambda_f}(1 - e^{-\Lambda_f})/N$ on the right side of equation (24) also agrees with the variance of the dry bundle theory. Therefore, the second term

$-3\sigma^2 e^{-\Lambda_{N-1}}(1-e^{-\Lambda_{N-1}})/4N$ must be a 'contribution' term brought from the effect of stress recovery in broken fibers. Since both the terms $e^{-\Lambda_{N-1}}$ and $(1 - e^{-\Lambda_{N-1}})$ range from 0 to 1, the second term must be zero or negative as follows:

$$-\frac{3}{4}\frac{\sigma^2}{N}e^{-\Lambda_{N-1}}\left(1 - e^{-\Lambda_{N-1}}\right) \leqslant 0. \tag{25}$$

Therefore, the variance $V\{\sigma_n\}/V_f^2$ is equal to or less than the variance of a dry bundle, also proving that stress recovery behavior due to sliding resistance is an effective mechanism for decreasing the variance of the composite stress. The expected value and variance analyzed here, equations (23) and (24), surprisingly, agree with the analysis results of Phoenix and Raj [9], i.e. equations (10) and (31) in their paper, despite the fact that the modeling in this paper is obviously different from theirs. From equations (24) the standard deviation $\sqrt{V\{\sigma_n\}}$ of σ_n is given as:

$$\sqrt{V\{\sigma_n\}} = V_f\left\{\frac{1}{2}\frac{\sigma}{\sqrt{N}}\sqrt{e^{-\Lambda_f}(1 - e^{-\Lambda_f})}\right\}. \tag{26}$$

The standard deviation is decreased in proportion to the inverse number of $\sqrt{N}$. The nature of this decrease is identical to that of the dry bundle theory. It should be noted in equation (26) that the standard deviation is reduced to $1/2$ due to the 'contribution'. Thus, stress recovery in broken fibers due to interfacial sliding resistance is a substantial mechanism for increasing the strength and reliability of composites.

The expected value and standard deviation of a dry bundle have been discussed from a statistical viewpoint and related to a normal distribution due to their binominal distribution and asymptotic nature [20]. We have proven, however, that the Markov process model proposed here can estimate the identical expected value and standard deviation of a unit fiber bundle from original definitions.

3.2. Maximum average stress

Figure 6 shows the relation between the normalized expected value $E\{\sigma_n\}/V_f\sigma_c$ and normalized fiber stress σ/σ_c, predicted from equation (23). The expected value indicates a maximum stress around $\sigma/\sigma_c = 0.88$. When this stress or more is applied to the composite, the composite may be fractured. It is difficult to analytically estimate the maximum stress $E\{\sigma_{n,\max}\}$ and its corresponding variance $V\{\sigma_{n,\max}\}$ from equations (23) and (24). Thus, we propose here a simple approximation method for estimating $E\{\sigma_{n,\max}\}$ and $V\{\sigma_{n,\max}\}$. The fiber stress corresponding to $E\{\sigma_{n,\max}\}$, denoted as σ^*, satisfies $\mathrm{d}E\{\sigma_n\}/\mathrm{d}\sigma = 0$, which is obtained by differentiating equation (23) with respect to σ, as follows:

$$\left(1 - m\Lambda_f\right)e^{-\Lambda_f} + 1 = 0. \tag{27}$$

This is rewritten as,

$$m\Lambda_f - 1 = e^{\Lambda_f}. \tag{27'}$$

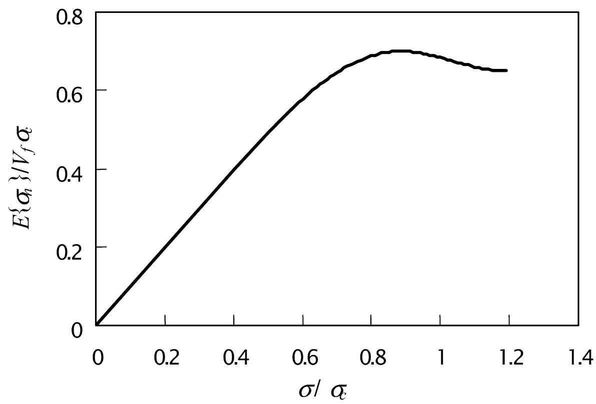

Figure 6. Normalized stress-strain diagram of a ceramic matrix composite at $m = 4$.

Table 1.
Expected values and standard deviations in strength of a ceramic matrix composite with various Weibull moduli

m	1	2	3	4	5	10	15	20
σ^*/σ_c	2.479	1.129	0.950	0.893	0.871	0.865	0.881	0.895
Hui *et al.* [13]	1.549	1.036	0.920	0.881	0.865	0.866	0.883	0.897
$E\{\sigma_{n,\max}\}/V_f\sigma_c$	1.242	0.698	0.685	0.700	0.717	0.786	0.827	0.853
$SD\sqrt{N}/V_f\sigma_c$	0.057	0.240	0.236	0.221	0.208	0.167	0.145	0.130

The right-hand of equation (27′) can be approximated as a linear equation, because the nonlinearity of the exponential function is not so significant when σ/σ_c is in the range of 0 to 1. We thus have

$$m\Lambda_f - 1 \approx a\Lambda_f + b \quad (a, b\text{: constants}). \tag{28}$$

The corresponding stress σ^* is estimated from equation (28) in the range of 0 to 1 as:

$$\sigma^* \approx \sigma_\mathrm{c}\left(\frac{1.8781}{m - 0.6944}\right)^{1/(m+1)}. \tag{29}$$

Table 1 shows σ^*/σ_c estimated for various Weibull moduli. On the other hand, Hui *et al.* [13], proposed the following approximation for σ^*/σ_c:

$$\sigma^*/\sigma_c = \{(2/m)(4m+2)/(4m+1)\}^{1/(m+1)}. \tag{30}$$

According to Hui *et al.* [13], this approximation agrees closely with the exact solution. The values approximated by equation (30) are also indicated in the table. The estimated σ^*/σ_c ratio obtained from equation (29) agrees relatively well with Hui's approximation except for $m = 1$. Since the Weibull modulus in strength of ceramic fibers generally ranges from 2 to 10 [22, 23], the linear approximation in equation (28) is concluded to be valid. In Table 1, $E\{\sigma_{n,\max}\}/V_f\sigma_c$

Table 2.
Expected values and standard deviations in strength of a ceramic matrix composite predicted by Markov process model ($m = 4$)

$\sigma_{\max}/\sigma_c$	$E\{\sigma_{n,\max}\}/V_f\sigma_c$	$SD\sqrt{N}/V_f\sigma_c$			
		$N = 10$	$N = 100$	$N = 1000$	$N = 10\,000$
0.886	0.700	6.916×10^{-2}	2.187×10^{-2}	6.916×10^{-3}	2.187×10^{-3}

and $SD\sqrt{N}/V_f\sigma_c$ are additionally shown. $E\{\sigma_{n,\max}\}/V_f\sigma_c$ is lower than σ^*/σ_c, but approaches σ^*/σ_c with an increase in m. $SD\sqrt{N}/V_f\sigma_c$ decreases dependent on m, except for $m = 1$. Table 2 shows $E\{\sigma_{n,\max}\}/V_f\sigma_c$ and $V\{\sigma_{n,\max}\}/V_f^2\sigma_c^2$, when $m = 4$ and $N = 10, 100, 1000$ and $10\,000$. As is easily seen from equations (23) and (24), while the average maximum stress is constant, the variance decreases in inverse proportion to the number of fibers N.

4. DISTRIBUTION FUNCTION FOR COMPOSITE STRENGTH

As in the assumption (2), the present model is composed of the chain-of-bundles, a series of unit fiber bundles, each of which is assumed to be statistically and structurally independent. The bundle length may be a normalized length δ_c, as defined in equation (10). When the composite has a length L, the number of the bundles M may be expressed as L/δ_c. The composite strength is then determined by the weakest bundle, when this bundle reaches its maximum stress $\sigma_{n,\max}$. As mentioned earlier, the structure of a chain-of-bundles allows strain localization, because each of unit fiber bundles would statistically have a different number of fiber breaks during loading. Curtin [11, 12] indicated analytically and through Monte-Carlo simulation that strain localization appears significantly only after the maximum strength is achieved. Therefore, this study assumes a chain-of-bundles without the influence of strain localization. According to the weakest link rule, the distribution function $H_{MN}(\sigma_{n,\max})$ of the composite strength is given as [9, 17, 18]

$$H_{MN}(\sigma_{n,\max}) = 1 - \{1 - G_N(\sigma_{n,\max})\}^M, \tag{31}$$

where $G_N(\sigma_{n,\max})$ is the distribution function of a unit fiber bundle.

How, then, will the distribution function $G_N(\sigma_{n,\max})$ be expressed? This has been discussed from the viewpoint of expressing a distribution function for bundle strength [9]. The present study verifies that the composite stress of equation (22), together with the effect of stress recovery, also behaves as a normal distribution. Let r denote the number of intact fibers at a stress σ, i.e. $r = N - i$. The random variable r has a binominal distribution P of order N, given as:

$$P = \binom{N}{r} p^r (1-p)^{N-r}, \tag{32}$$

where, $p = \exp(-\Lambda_f)$. The expected number of intact fibers and the variance of the binominal distribution are given as

$$E\{r\} = Np, \tag{33}$$

$$V\{r\} = Np(1-p). \tag{34}$$

When $N \gg 1$, the binominal distribution approaches a normal distribution as follows

$$\begin{aligned} f_B(r)\mathrm{d}r &= \frac{1}{\sqrt{2\pi Np(1-p)}}\exp\left\{-\frac{(r-Np)^2}{2Np(1-p)}\right\}\mathrm{d}r \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z \equiv \phi(z)\mathrm{d}z, \end{aligned}$$

where,

$$z = \frac{r-Np}{\sqrt{Np(1-p)}} \tag{35}$$

f_B is the density of r. $\phi(z)$ is the standard normal density. Its cumulative distribution is

$$\Phi(z) = \int_{-\infty}^{z}\phi(x)\,\mathrm{d}x. \tag{36}$$

Equation (22) is changed as follows:

$$\begin{aligned} \sigma_n &= V_f\left\{\frac{N-i}{N}\sigma + \frac{i}{N}\frac{\sigma}{2}\right\} = V_f\frac{\sigma}{2}\left(\frac{r}{N}+1\right), \\ \because r &= N\left(\frac{\sigma_n}{\sigma}\frac{2}{V_f}-1\right). \end{aligned} \tag{37}$$

Substitution of equation (38) into equation (36) gives

$$\begin{aligned} z &= \frac{r-Np}{\sqrt{Np(1-p)}} \\ &= \frac{N\left(\frac{\sigma_n}{\sigma}\frac{2}{V_f}-1\right)-Np}{\sqrt{Np(1-p)}} = \frac{\sigma_n - V_f\left(\frac{\sigma}{2}p+\frac{\sigma}{2}\right)}{V_f\frac{\sigma}{2}\sqrt{p(1-p)/N}} = \frac{\sigma_n - V_f\left\{\sigma p + \frac{\sigma}{2}(1-p)\right\}}{V_f\frac{\sigma}{2}\sqrt{p(1-p)/N}} \\ &= \frac{\sigma_n - E\{\sigma_n\}}{\sqrt{V\{\sigma_n\}}}. \end{aligned} \tag{38}$$

Thus, the asymptotic nature of the normal distribution of the number of intact fibers, $N-i$, is also inherited by the composite stress. Therefore, $G_N(\sigma_{n,\max})$ can be given as:

$$G_N\left(\sigma_{n,\max}\right) = \frac{1}{\sqrt{2\pi V\left\{\sigma_{n,\max}\right\}}} \int_{-\infty}^{\sigma_{n,\max}} \exp\left\{\frac{1}{2}\left(\frac{\sigma_{n,\max} - E\left\{\sigma_{n,\max}\right\}}{\sqrt{V\left\{\sigma_{n,\max}\right\}}}\right)^2\right\}. \quad (39)$$

Although equation (40) requires numerical evaluation, recent computer software can quickly support the numerical calculation. Figure 7 shows the distribution of the composite strength obtained from equations (31) and (40), when $M = 1$, 10, 100 and 10 000. Figure 7 uses the expected value and standard deviation at $m = 4$ shown in Table 2. The coordinate is normalized by the parameter $\sigma_{n,\max}/\sigma_c$. As seen in every figure, the composite strength is decreased with an increase in the number of partition M and decreases significantly when $N = 10$ due to the size effect from the large variation in strength of a unit fiber bundle of $N = 10$. As the number of fibers N is increased, the composite strength approaches 0.700, the expected value in strength for a unit fiber bundle. This means that the composite strength can be treated more deterministically if the number of fibers included in the composite increases. Since the number of fibers included in practical composites realistically exceeds 10 000, it is concluded that the ceramic matrix composite is theoretically reliable in strength.

5. CONCLUSION

This paper proposed a strength reliability model for a unidirectional fiber-reinforced ceramic matrix composite, based on a Markov process model, in order to predict, stochastically, a damage state of the composite during loading. The major conclusions are summarized below:

(1) The probabilities of being in fiber breakage states were analytically solved from the state transition diagram in which the number of broken fibers was expressed as a discrete set of states. Using the solutions of the probabilities, reliable fiber stress levels for damage tolerant design of the composites were predicted, from the viewpoint of materials reliability engineering. Furthermore, it was shown that the composite's fracture stress was given deterministically when the number of fibers is equal to or more than fifty.

(2) To compare this stochastic process analysis with previously proposed strength models in the basis of classical bundle theory, the expected value in stress and the variance were analytically solved from the probabilities of being in the states in the above. The expected value and variance are both composed of two terms, which are equivalent to the effects of a dry bundle and stress recovery in broken fibers. The effect of stress recovery takes a positive value in the expected value and a negative value in the variance. These solutions agree completely with the solutions obtained by Phoenix and Raj [9]. In addition, the expected value of the composite strength and its variance were predicted

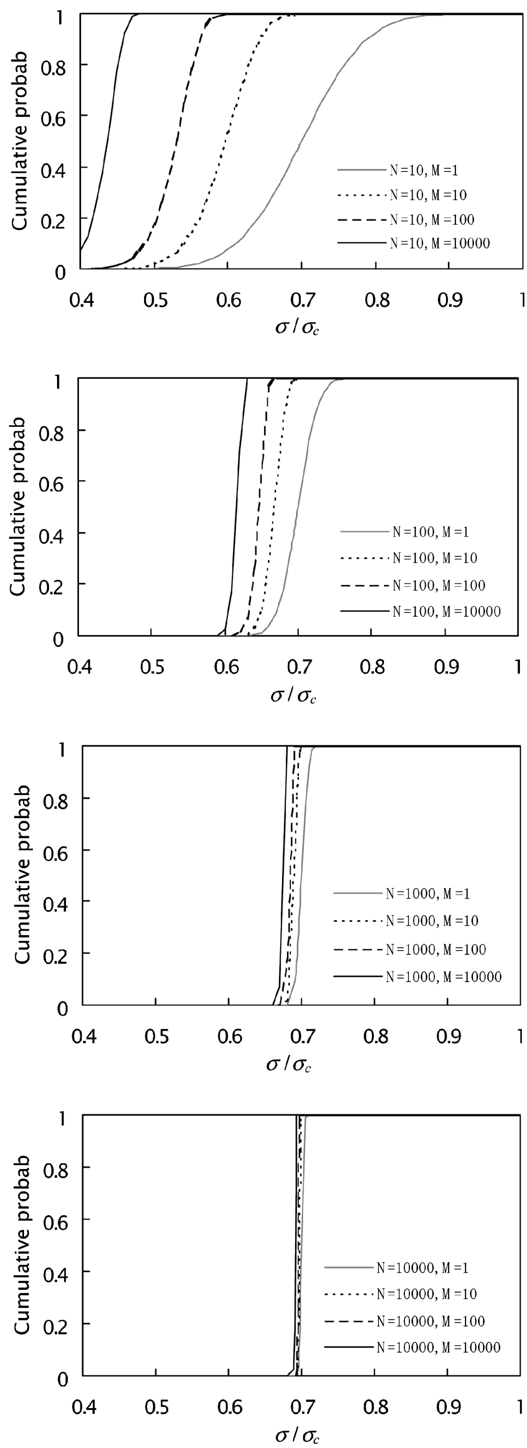

Figure 7. Cumulative distributions for a ceramic matrix composite with various sizes.

from the solutions. The corresponding value of normalized fiber stress required to obtain the expected value in strength agreed relatively well with the value predicted by Hui *et al*. [13].

(3) We further verified that the composite strength obeys a normal distribution, which has the expected value and standard deviation predicted in the above. The normal distribution can be obtained from an asymptotic form of the binominal distribution function, similarly to the procedure in a dry bundle model, since the normal distribution was assumed as a composite structure. According to this model, if the number of fibers approaches the number in practical composites, the ceramic matrix composite will become a reliable material without variation in strength.

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APPENDIX 1

The term $\sum_{i=0}^{N-1}(N-i)P_i$ of equation (19) is changed as follows:

$$\begin{aligned}
\sum_{i=0}^{N-1}(N-i)P_i &= \sum_{i=0}^{N-1}(N-i)(-1)^i\binom{N}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\exp\left(-\Lambda_j\right) \\
&= \sum_{j=0}^{N-1}(-1)^j\exp(-\Lambda_j)\sum_{i=j}^{N-1}(-1)^i(N-i)\binom{N}{i}\binom{i}{j} \\
&= \sum_{j=0}^{N-1}(-1)^j(N-j)\binom{N}{j}\exp(-\Lambda_j)\underbrace{\sum_{i=j}^{N-1}(-1)^i\binom{N-j-1}{i-j}}_{1} \\
&= \sum_{j=0}^{N-1}(-1)^i(N-j)\binom{N}{j}\exp(-\Lambda_j)\underbrace{\sum_{i=0}^{N-j-1}(-1)^i\binom{N-j-1}{i}}_{2} \\
&= (-1)^{N-1}\cdot 1\cdot\binom{N}{N-1}\exp(-\Lambda_{N-1})\cdot(-1)^{N-1}. \qquad \text{(A-1)}
\end{aligned}$$

In general, $\sum_{i=0}^{k}(-1)^i\binom{k}{i}=0$ for $k \geqslant 1$ (k is a positive integer). Therefore, the underlined term **2** is equal to zero except for $j = N-1$. On the other hand, when $j = N-1$, the underlined term **1** is equal to $(-1)^{N-1}$.

$$\begin{aligned}
\sum_{i=0}^{N-1}(N-i)P_i &= N\exp\left(-\Lambda_{N-1}\right) \\
&= N\exp(-\Lambda_f). \qquad \text{(A-2)}
\end{aligned}$$

Equation (A-1) is the expected value of the number of intact fibers, $E\{N-i\}$.

APPENDIX 2

The term $\sum_{i=0}^{N}(N-i)^2 P_i$ of equation (20) is changed as follows:

$$\begin{aligned}
&\sum_{i=0}^{N}(N-i)^2 P_i \\
&\quad = \sum_{i=0}^{N-1}(N-i)^2 P_i \\
&\quad = \sum_{i=0}^{N-1}(N-i)^2 (-1)^i \binom{N}{i} \sum_{j=0}^{i}(-1)^j \binom{i}{j} \exp\left(-\Lambda_j\right) \\
&\quad = \sum_{j=0}^{N-1}(-1)^j \exp\left(-\Lambda_j\right) \sum_{i=j}^{N-1}(-1)^i (N-i)^2 \binom{N}{i}\binom{i}{j} \\
&\quad = \sum_{j=0}^{N-1}(-1)^j (N-j)\binom{N}{j}\exp(-\Lambda_j) \sum_{i=j}^{N-1}(-1)^i (N-i)^2 \binom{N-j-1}{i-j} \\
&\quad = \sum_{j=0}^{N-1}(-1)^j (N-j)\binom{N}{j}\exp(-\Lambda_j) \underbrace{\sum_{i=0}^{N-j-1}(-1)^i (N-j-i)\binom{N-j-1}{i}}_{\mathbf{3}}.
\end{aligned} \tag{A-3}$$

The underlined term **3** is separated as:

$$\begin{aligned}
&= \sum_{i=0}^{N-j-1}(-1)^i (N-j)\binom{N-j-1}{i} - \sum_{i=0}^{N-j-1}(-1)^i i\binom{N-j-1}{i} \\
&= (N-j)\sum_{i=0}^{N-j-1}(-1)^i \binom{N-j-1}{i} + \sum_{i=1}^{N-j-1}(-1)^{i-1} i\binom{N-j-1}{i}.
\end{aligned} \tag{A-4}$$

In general, $\sum_{i=1}^{k}(-1)^{i-1} i\binom{k}{i} = 0$ for $k \geqslant 2$. Therefore, the term **3** is equal to zero when $j \leqslant N-3$, and equal to $(-1)^{N-2}$ and $(-1)^{N-1}$ when $j = N-2$ and $j = N-1$, respectively. Thus,

$$\begin{aligned}
\sum_{i=0}^{N}(N-i)^2 P_i &= (-1)^{N-2} N(N-1)\exp\left(-\Lambda_{N-2}\right) + N\exp\left(-\Lambda_{N-1}\right) \\
&= N\exp(-\Lambda_{N-1})\{(N-1)\exp(-\Lambda_{N-1}) + 1\},
\end{aligned} \tag{A-4}$$
$$(\because \exp(-\Lambda_{N-2}) = \exp(-2\Lambda_{N-1})).$$

APPENDIX 3

Equation (23) is rewritten as:

$$\begin{aligned} E\{\sigma_n\} &= V_f \sum_{i=0}^{N} \left(\frac{N-i}{N}\sigma + \frac{i}{N}\frac{\sigma}{2} \right) P_i \\ &= V_f \left(\frac{1}{2} \sum_{i=0}^{N} \frac{N-i}{N}\sigma P_i + \sum_{i=0}^{N} \frac{\sigma}{2} P_i \right) \\ &= V_f \left(\frac{1}{2} E\left\{\sigma_n^{(b)}\right\} + \frac{\sigma}{2} \right), \end{aligned} \tag{A-5}$$

where $\sigma_n^{(b)}$ is a stress of dry bundle without the term contributed from stress recovery, defined as $\sigma_n^{(b)} = (N-i)\sigma/N$. $E\{\sigma_n^{(b)}\}$ is the expected value of the bundle stress, given as:

$$E\left\{\sigma_n^{(b)}\right\} = \sum_{i=0}^{N} \frac{N-i}{N}\sigma P_i = \frac{\sigma}{N}\sum_{i=0}^{N-1}(N-i)P_i = \sigma \exp(-\Lambda_f). \tag{A-6}$$

Substitution of equation (A-6) into equation (A-5) yields

$$\begin{aligned} E\{\sigma_n\} &= V_f \left(\frac{\sigma}{2} e^{-\Lambda_f} + \frac{\sigma}{2} \right) \\ &= V_f \left\{ \sigma e^{-\Lambda_f} + \frac{\sigma}{2}\left(1 - e^{-\Lambda_f}\right) \right\}. \end{aligned} \tag{A-7}$$

APPENDIX 4

As in the procedure of Appendix 3, the variance of a bundle is first found, as follows:

$$\begin{aligned} V\left\{\sigma_n^{(b)}\right\} &= \sum_{i=0}^{N-1} \left(\frac{N-i}{N}\sigma \right)^2 P_i - \left(E\left\{\sigma_n^{(b)}\right\}\right)^2 \\ &= \frac{\sigma^2}{N^2} \sum_{i=0}^{N-1} (N-i)^2 P_i - \left(E\left\{\sigma_n^{(b)}\right\}\right)^2 \\ &= \frac{\sigma^2}{N^2} \cdot N e^{-\Lambda_{N-1}} \left\{ (N-1) e^{-\Lambda_{N-1}} + 1 \right\} - \left(\sigma e^{-\Lambda_{N-1}}\right)^2 \\ &= \frac{\sigma^2}{N} \left\{ (N-1)\left(e^{-\Lambda_{N-1}}\right)^2 + e^{-\Lambda_{N-1}} - N\left(e^{-\Lambda_{N-1}}\right)^2 \right\} \\ &= \frac{\sigma^2}{N} e^{-\Lambda_{N-1}} \left(1 - e^{-\Lambda_{N-1}}\right). \end{aligned} \tag{A-8}$$

The variance $V\{\sigma_n\}$ is obtained, by substituting equation (A-8) into equation (15), as

$$\begin{aligned}
V\{\sigma_n\} &= \sum_{i=0}^{N} V_f^2 \left(\frac{N-i}{N}\sigma + \frac{i}{N}\frac{\sigma}{2}\right)^2 P_i - (E\{\sigma_n\})^2 \\
&= \sum_{i=0}^{N} V_f^2 \left(\frac{\sigma_n^{(b)}}{2} + \frac{\sigma}{2}\right)^2 P_i - (E\{\sigma_n\})^2 \quad \text{(See, equation (A-5))} \\
&= \frac{V_f^2}{4}\left[\sum_{i=0}^{N}\left(\sigma_n^{(b)}\right)^2 P_i + 2\sigma\sum_{i=0}^{N}\sigma_n^{(b)} P_i + \sigma^2\sum_{i=0}^{N} P_i \right. \\
&\qquad \left. - 4\left\{\sigma e^{-\Lambda_{N-1}} - \frac{\sigma}{2}\left(1 - e^{-\Lambda_{N-1}}\right)\right\}^2\right] \\
&= \frac{V_f^2}{4}\left\{\sum_{i=0}^{N}\left(\sigma_n^{(b)}\right)^2 P_i - \left(\sigma e^{-\Lambda_{N-1}}\right)^2\right\} \\
&= V_f^2 \cdot \frac{1}{4} V\left\{\sigma_n^{(b)}\right\} \\
&= V_f^2\left\{\frac{\sigma^2}{N} e^{-\Lambda_{N-1}}\left(1 - e^{-\Lambda_{N-1}}\right) - \frac{3}{4}\frac{\sigma^2}{N} e^{-\Lambda_{N-1}}\left(1 - e^{-\Lambda_{N-1}}\right)\right\}. \qquad \text{(A-9)}
\end{aligned}$$